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Bäcklund transformations for the (un)pumped Maxwell–Bloch system and the fifth Painlevé equation

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Abstract. A Bäcklund transformation for the (un)pumped Maxwell–Bloch system governing phenomena in nonlinear optics is derived. The approach is based on the use of constant coefficient ideals (CC ideals). Since the Maxwell–Bloch system in a certain one-dimensional reduction is connected to the fifth Painlevé equation via a contact transformation, chains of solutions for the latter equation are obtained. As illustration, some rational solutions are explicitly given.

1. Introduction

During the past twenty years the study of nonlinear integrable systems has become more and more sophisticated. Against this background new interest has been shown in the construction of physically relevant solutions of the (un)pumped Maxwell–Bloch system of nonlinear optics (see [1] and references therein). In the absence of pumping it describes sharpline self-induced transparency and is equivalent to a system governing stimulated Raman scattering in the transient limit [2]. In this case the Maxwell–Bloch system can be regarded as compatibility condition for an ordinary Lax pair [3].

The pumped Maxwell–Bloch system, however, admits a non-isospectral linear problem (Lax pair) which is amenable to a generalized version of the inverse scattering method. In particular, soliton solutions have been obtained using this technique [4]. Here, it is shown that the (un)pumped Maxwell–Bloch system can be represented by a 'CC ideal' (a closed set of differential two-forms with constant coefficients) or 'invariant differential system' as introduced by Harrison [5] and Estabrook [6], respectively. In subsequent papers Hoenselaers [7, 8] has taken up this notion and derived CC ideals for various integrable equations including the classical sine–Gordon equation, (modified) Korteweg–de Vries equation and nonlinear Schrödinger equation.

CC ideals may be used to generate Bäcklund transformations for the underlying nonlinear equations as presented in [9]. It turns out that an extended approach is applicable to the 'non-isospectral' CC ideal of the Maxwell–Bloch system. Moreover, this Bäcklund transformation is compatible with a certain similarity reduction of the Maxwell–Bloch system leading to an ordinary differential equation of second order which is related to the fifth Painlevé equation via a contact transformation [10]. The Bäcklund transformation can be iterated and therefore generates hierarchies of solutions for one of the Painlevé equations which play an important role in the singularity manifold analysis [1].

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2. The CC ideal

In two papers on prolongation structures for nonlinear partial differential equations [7, 8] Hoenselaers has demonstrated extensively how to derive differential equations from constant coefficient ideals (CC ideals) or invariant differential systems originally expounded by Harrison [5] and Estabrook [6]. This procedure may be regarded as the reverse of the Wahlquist–Estabrook prolongation technique [11, 12] and reads in the present context as follows:

The prolongation algebra in question is the semi-direct sum of the infinite-dimensional loop algebra of $sl(2, R)$ (essentially the Kac–Moody algebra $A_1^{(1)}$ [13]) and the Virasoro algebra, viz

$$sl(2, R) \otimes R(\lambda, \lambda^{-1}) \oplus \mathit{Virasoro}. \tag{1}$$

It obeys the commutator relations

$$[X_i^n, X_j^m] = [X_i, X_j]^{n+m} \qquad [D^n, X_i^m] = mX_i^{n+m} \qquad [D^n, D^m] = (m-n)D^{n+m} \tag{2}$$

where the simple Lie algebra $sl(2, R)$ is defined via the relations

$$[X_1, X_2] = -2X_2 \qquad [X_1, X_3] = 2X_3 \qquad [X_2, X_3] = X_1\,. \tag{3}$$

In order to construct a CC ideal for the (un)pumped Maxwell–Bloch system we shall focus on the basic generators $X_1^{-1}, X_2^0, X_3^0, X_1^1, X_2^1, X_3^1$ and D^2. Accordingly (cf, e.g. [8] on how this formalism works), we can introduce one-forms

$$\begin{array}{lll} \xi_{-1}^1 \text{ dual to } X_1^{-1} & \xi_0^2 \text{ dual to } X_2^0 & \xi_0^3 \text{ dual to } X_3^0 \\ \xi_1^1 \text{ dual to } X_1^1 & \xi_1^2 \text{ dual to } X_2^1 & \xi_1^3 \text{ dual to } X_3^1 \\ \eta_2 \text{ dual to } D^2 & & \end{array}$$

which give rise to a closed ideal of differential two-forms generated† by the differential 'ρ system'

$$\begin{aligned} &\mathrm{d}\eta_2 = 0 \\ &\mathrm{d}\xi_{-1}^1 = 0 \qquad \mathrm{d}\xi_0^2 = -2\xi_{-1}^1\,\xi_1^2 \qquad \mathrm{d}\xi_0^3 = 2\xi_{-1}^1\,\xi_1^3 \\ &\mathrm{d}\xi_1^1 = -\eta_2\,\xi_{-1}^1 + \xi_0^2\,\xi_1^3 + \xi_1^2\,\xi_0^3 \qquad \mathrm{d}\xi_1^2 = -2\xi_1^1\,\xi_0^2 \qquad \mathrm{d}\xi_1^3 = 2\xi_1^1\,\xi_0^3 \end{aligned} \tag{4}$$

and the algebraic 'σ system'

$$\begin{aligned} &\eta_2\,\xi_1^1 = \eta_2\,\xi_1^2 = \eta_2\,\xi_1^3 = 0 \\ &\xi_{-1}^1\,\xi_0^2 = \xi_{-1}^1\,\xi_0^3 = \xi_0^2\,\xi_0^3 = 0 \\ &\xi_1^1\,\xi_1^2 = \xi_1^1\,\xi_1^3 = \xi_1^2\,\xi_1^3 = 0\,. \end{aligned} \tag{5}$$

The fact that the structure constants of the Lie algebra (1) satisfy the Jacobi identities guarantees that the (ρ, σ) system is closed under exterior differentiation. If we were dealing with the complete set of generators of a finite-dimensional Lie algebra the σ system would not be present and the ρ system just constitute the Maurer–Cartan structure forms of that algebra.

† Hereafter we shall commit the usual impropriety in referring to the system (4) as 'closed ideal'. Furthermore, the wedge between differential forms will be omitted.

As an additional constraint we demand that the one-forms $\eta_2, \xi^1_{-1}, \xi^1_1$ be real and ξ^2_0, ξ^3_0 and $\xi^2_1, -\xi^3_1$ form complex conjugate pairs respectively. We note that this condition is compatible with the (ρ, σ) system (4), (5). Having this in mind we can now look for exact one-forms within the ρ system to be used as coordinate differentials on the guaranteed existing integral manifolds (Cartan's calculus of exterior differential forms [14]). The obvious choice is, of course,

$$\eta_2 = \tfrac{1}{2}c^2\mathrm{d}x \qquad \xi^1_{-1} = \tfrac{1}{2}\mathrm{d}t \tag{6}$$

with some constant c. The algebraic σ system implies that

$$\xi^1_1 \sim \xi^2_1 \sim \xi^3_1 \sim \eta_2 \qquad \xi^2_0 \sim \xi^3_0 \sim \xi^1_{-1}$$

and hence a parametrization of our one-forms may be introduced according to

$$\begin{aligned} &\xi^1_1 = \tfrac{1}{2}N\mathrm{d}x \qquad \xi^2_1 = \tfrac{1}{2}\rho^*\mathrm{d}x \qquad \xi^3_1 = -\tfrac{1}{2}\rho\mathrm{d}x \\ &\xi^2_0 = \tfrac{1}{2}E^*\mathrm{d}t \qquad \xi^3_0 = \tfrac{1}{2}E\mathrm{d}t \end{aligned} \tag{7}$$

where the numerical factors have been chosen for convenience and the asterisk denotes complex conjugation.

The final step in the procedure is to insert the one-forms ξ^i_n and η_2 as given by (6) and (7) into the remaining equations of the ρ system (4) which produces the first-order system

$$\rho_t = NE \tag{8a}$$

$$E_x = \rho \tag{8b}$$

$$N_t + \tfrac{1}{2}(\rho E^* + \rho^* E) = \tfrac{1}{2}c^2. \tag{8c}$$

The above system constitutes the pumped Maxwell–Bloch system descriptive of the propagation of radiation pulses in a two level atomic system. The quantities involved are the complex envelope E, the population N and the polarization ρ. The pumping of atoms between the ground and excited state is characterized by the constant c.

Interestingly, the (un)pumped Maxwell–Bloch system can be brought into (at least) two equivalent forms which provides links to other known integrable equations. Firstly, on setting

$$E = 2S \qquad \rho = 2A_1A_2^* \qquad N = |A_1|^2 - |A_2|^2$$

the unpumped Maxwell–Bloch system ($c = 0$) transforms into

$$A_{1t} = -SA_2 \qquad A_{2t} = S^*A_1 \qquad S_x = A_1A_2^* \tag{9}$$

where $|A_1|^2 + |A_2|^2 = 1$ has been assumed without loss of generality [2]. The above system represents the stimulated Raman scattering equations under certain assumptions. (It should be emphasized that here, as well as in the case of the Maxwell–Bloch system, x and t are in general not the physical variables.)

Secondly, multiplication of (8c) by N and use of (8a) yields

$$(N^2 + \rho\rho^*)_t = c^2N$$

which suggests parametrizing N and ρ in terms of trigonometric/exponential functions. In fact, putting

$$N = a\cos\theta \qquad \rho = -a\sin\theta\mathrm{e}^{\mathrm{i}\varphi} \tag{10}$$

we obtain

$$\begin{aligned} &\theta_{tx} + \tfrac{1}{2}c^2 \left(\frac{\sin\theta}{a}\right)_x - \varphi_t\varphi_x \tan\theta = a\sin\theta \\ &(\varphi_t \tan\theta)_x + \left(\theta_t + \tfrac{1}{2}c^2\frac{\sin\theta}{a}\right)\varphi_x = 0 \qquad (11) \\ &a_t = \tfrac{1}{2}c^2\cos\theta\,. \end{aligned}$$

The above equations may be termed 'deformed self-induced transparency equations' for different reasons. On the one hand, for $c = 0$, they describe sharp-line, self-induced transparency as mentioned in the introduction. On the other hand, in the limit $\varphi = 0$, a particular case of the deformed sine–Gordon system defining hyperbolic surfaces in differential geometry (Bianchi surfaces) is obtained [15, 16]. Here, the negative Gaussian curvature of the surfaces depends only on t. We shall see later that the Bäcklund transformation to be derived is in agreement with the one associated with surfaces of constant negative curvature, i.e. the classical Bäcklund transformation for the sine–Gordon equation.

3. The prolongation structure

For the derivation of a Bäcklund transformation for the Maxwell–Bloch system it is now necessary to set up a (linear) problem for the CC ideal (4), (5). To this end let us suppose that the Lie algebra (1) is realized as vectorfields living on a manifold labelled by the components of some vector-valued variable y, say. In this case the commutator appearing in (2) is the usual Lie bracket (up to the sign) between vectorfields with respect to the 'pseudopotentials' y, viz

$$[Y, Z]^\alpha = Y^\alpha_{,\beta}Z^\beta - Z^\alpha_{,\beta}Y^\beta$$

(Einstein's summation convention!)†. Then, by construction, the set of one-forms

$$\Omega = -\mathrm{d}y + X^n_i\xi^i_n + D^2\eta_2 \qquad (12)$$

is closed, i.e. $\mathrm{d}\Omega = 0 \bmod \Omega$, iff the one-forms ξ^i_n and η_2 satisfy the (ρ, σ) system. This condition, in turn, guarantees that the equations $\Omega = 0$ are integrable [11, 12].

More specifically, in terms of the standard representation of the Lie algebra (1)

$$(X^n_i)^\alpha\partial_{y^\alpha} = \lambda^n\langle X_i\phi, \partial_\phi\rangle \qquad (D^n)^\alpha\partial_{y^\alpha} = -\lambda^{n+1}\partial_\lambda$$

the set of one-forms (12) splits into

$$\Omega^L = -\mathrm{d}\phi + \lambda^n X_i\xi^i_n\phi \qquad (13)$$

$$\Omega^V = -\mathrm{d}\lambda - \lambda^3\eta_2\,. \qquad (14)$$

Here, the pseudopotential vector y consists of the components $(\phi^1, \phi^2, \lambda)$, $\langle X_i\phi, \partial_\phi\rangle$ denotes the scalar product between those two-dimensional vectors and the matrices X_i constitute the standard faithful and tracefree representation of the $sl(2, R)$ algebra as given by (3), namely

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \qquad (15)$$

† Note that throughout this paper greek indices label components of objects, whereas roman indices number the objects themselves.

Furthermore, restriction of Ω to an integral manifold

$$\Omega|_{y=y(x,t)} = 0$$

and use of the explicit parametrization (6), (7) produces the Frobenius system

$$\phi_x = \frac{\lambda}{2}\begin{pmatrix} N & -\rho \\ -\rho^* & -N \end{pmatrix}\phi \tag{16}$$

$$\phi_t = \left[\frac{1}{2}\begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix} + \frac{\lambda^{-1}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\phi \tag{17}$$

together with

$$\lambda_x = -\tfrac{1}{2}c^2\lambda^3 \qquad \lambda_t = 0\,. \tag{18}$$

The linear problem (16), (17) is non-isospectral, as pointed out in [10], since the 'parameter' λ depends on the coordinate x. In fact, the first-order system (18) can be integrated, giving

$$\lambda = \frac{1}{c\sqrt{x+K}} \tag{19}$$

where K is a real integration constant. It is evident that the integrability condition for the Frobenius system (16)–(18) is satisfied modulo the Maxwell–Bloch system (8*a*).

4. The Bäcklund transformation

The following approach is based on the 'dressing' method developed by Neugebauer and Kramer which provides Bäcklund transformations for nonlinear equations given as compatibility condition of (non-)isospectral Lax pairs of the form (16)–(18). This 'N-soliton ansatz' had its origin in the study of Ernst's equation of general relativity in 1979 [17]. Subsequently, it has been applied to Einstein–Maxwell fields in general relativity and the AKNS system [18, 19]. A generalization of this method to CC ideals has been discussed in [9] the essence of which may be summarized as follows.

Theorem 1. Let a tracefree 2×2-matrix-valued one-form $X(\lambda)$ be polynomial of degree k in a constant λ and degree l in λ^{-1}. Furthermore, let ϕ be a two-dimensional pseudopotential vector and

$$P(\lambda) = \sum_{n=0}^{N_0} P_n\lambda^n \in R^{2,2}$$

be restricted by

(i) $$\det P_{N_0} = \text{constant} \neq 0 \tag{20}$$

(ii) $$P(\lambda_r)\phi_r = 0 \tag{21}$$

for arbitrary but distinct constants λ_r, $r = 1,\ldots,2N_0$, and linearly independent pseudopotential vectors ϕ_r.

Then the vector-valued one-form

$$\Omega = -d\phi + X(\lambda)\phi \tag{22}$$

is form-invariant under

$$\phi \to \phi' = P(\lambda)\phi \tag{23}$$

$$\Omega \to \Omega' = P(\lambda)\Omega \tag{24}$$

$$X(\lambda) \to X'(\lambda) = P(\lambda)X(\lambda)P^{-1}(\lambda) + \mathrm{d}P(\lambda)P^{-1}(\lambda) \tag{25}$$

where the last relation has to be taken modulo $\Omega_r = \Omega|_{\phi=\phi_r,\lambda=\lambda_r}$.

In order not to clutter up the formulae with too many symbols we have confined ourselves to the relevant case of two-dimensional objects. The condition (20) guarantees that the new one-form $X'(\lambda)$ is again tracefree whereas the condition (21) preserves the polynomial structure and the degree of $X'(\lambda)$. Hence, $X'(\lambda_r)$ is understood as $\lim_{\lambda\to\lambda_r} X'(\lambda)$. It should be stressed that (21) constitutes a linear algebraic system which determines $P(\lambda)$ up to one coefficient P_{n_0}, say.

Unfortunately, theorem 1 in the present form is not appropriate for the purposes pursued in this paper since λ is treated as a constant. Consequently, the above theorem provides a Bäcklund transformation only for CC ideals based on the loop algebra of $sl(2, R)$. Nevertheless, the formal structure of the one-form Ω^L (13) and (22) is the same though λ therein is a pseudopotential itself. The following modification of theorem 1 takes this fact into account.

Theorem 2. Let a tracefree 2×2-matrix-valued one-form $X(\lambda)$ be polynomial of degree k in a pseudopotential λ and degree l in λ^{-1}. Furthermore, let a one-form $D(\lambda)$ be polynomial of degree $k+2$ in λ and l in λ^{-1} and

$$P(\lambda) = f(\lambda)Q(\lambda) = \prod_{r=1}^{2N_0}(\lambda-\lambda_r)^{-1/2}\sum_{n=0}^{N_0} Q_n\lambda^n \in R^{2,2}$$

be restricted by

(i) $$\det Q_{N_0} = \text{constant} \neq 0 \tag{26}$$

(ii) $$Q(\lambda_r)\phi_r = 0 \tag{27}$$

for arbitrary but distinct pseudopotentials λ_r, $r = 1, \ldots, 2N_0$, and linearly independent pseudopotential vectors ϕ_r.

Then the vector-valued one-form

$$\Omega^L = -\mathrm{d}\phi + X(\lambda)\phi$$

is form-invariant under

$$\begin{aligned}
&\phi \to \phi' = P(\lambda)\phi \\
&\Omega^L \to \Omega^{L\prime} = P(\lambda)\Omega^L \\
&X(\lambda) \to X'(\lambda) = P(\lambda)X(\lambda)P^{-1}(\lambda) + \mathrm{d}P(\lambda)P^{-1}(\lambda)
\end{aligned} \tag{28}$$

whereas the one-form

$$\Omega^V = -\mathrm{d}\lambda - D(\lambda)$$

remains unchanged. Analogous to theorem 1, (28) has to be taken modulo Ω_r^L and Ω_r^V.

Proof. To prove theorem 2 it is necessary to show two properties of the transformed one-form $X'(\lambda)$. Firstly, since $\det Q(\lambda)$ is a polynomial of degree $2N_0$ in λ and condition (27) implies that $\det Q(\lambda_r) = 0$, we conclude that

$$\det Q(\lambda) \sim \prod_{r=1}^{2N_0} (\lambda - \lambda_r) .$$

Thus, (26) yields

$$\det P(\lambda) = \text{constant}$$

and consequently

$$\operatorname{Tr} X'(\lambda) = \operatorname{Tr} [\mathrm{d}P(\lambda)P^{-1}(\lambda)] = \mathrm{d}[\ln \det P(\lambda)] = 0 .$$

Secondly, it can readily be shown that the first two terms of the right-hand side of

$$X'(\lambda) = Q(\lambda)X(\lambda)Q^{-1}(\lambda) + \mathrm{d}Q(\lambda)Q^{-1}(\lambda) + \frac{\mathrm{d}f(\lambda)}{f(\lambda)} \tag{29}$$

form a (Laurent) polynomial in λ by following the proof of theorem 1 [9]. The remaining term

$$\frac{\mathrm{d}f(\lambda)}{f(\lambda)} = \frac{\mathrm{d}\prod_{r=0}^{2N_0}(\lambda - \lambda_r)^{-1/2}}{\prod_{r=0}^{2N_0}(\lambda - \lambda_r)^{-1/2}} = \frac{1}{2}\sum_{r=0}^{2N_0} \frac{D(\lambda) - D(\lambda_r)}{\lambda - \lambda_r}$$

is regular at the zeros λ_r since $D(\lambda)$ is a (Laurent) polynomial in λ. Moreover, on writing equation (28) in the form

$$X'(\lambda)Q(\lambda) = Q(\lambda)X(\lambda) + \mathrm{d}Q(\lambda) + \frac{\mathrm{d}f(\lambda)}{f(\lambda)}Q(\lambda) \tag{30}$$

it is easily proven that terms proportional to λ^{N_0+k+1} in (30) cancel out so that $X'(\lambda)$ is indeed a polynomial in λ of degree k and in λ^{-1} of degree l.

Now, the one-form Ω^L associated with the Maxwell–Bloch system can be characterized uniquely via the properties of

$$X(\lambda) = \mathcal{X}_n \lambda^n = \lambda^n X_i \xi^i_n$$

as given in (13)

$$X(\lambda) \text{ is a polynomial of degree } k = 1 \text{ and } l = 1, \text{ respectively} \tag{31}$$

$$\mathcal{X}_{-1} \text{ does } \textit{not} \text{ contain the generators } X_2 \text{ and } X_3 \tag{32}$$

$$X(-\lambda) = MX^*(\lambda)M^{-1} \tag{33}$$

with

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

For the new one-form $X'(\lambda)$ condition (31) is covered by theorem 2. Invariance of the remaining conditions (32) and (33) requires restrictions on Q_0 and the zeros λ_r according to the following theorem.

Theorem 3 (N_0-fold Bäcklund transformation for the Maxwell–Bloch CC ideal). The one-form $X'(\lambda) = \mathcal{X}'_n\lambda^n$ as given in theorem 2 satisfies (32) and (33) for

$$Q_0 = \prod_{r=0}^{N_0} \lambda_r 1 \tag{34}$$

and

$$\begin{aligned} \lambda_{N_0+r} &= -\lambda_r \qquad r = 1, \dots, N_0 \\ \phi_{N_0+r} &= M\phi_r \,. \end{aligned} \tag{35}$$

Evaluation of (30) leads to

$$\begin{aligned} \mathcal{X}'_{-1} &= \mathcal{X}_{-1} \\ \mathcal{X}'_0 &= Q_0\mathcal{X}_0Q_0^{-1} + [Q_1, \mathcal{X}_{-1}]Q_0^{-1} \\ \mathcal{X}'_1 &= Q_{N_0}\mathcal{X}_1Q_{N_0}^{-1} + \tfrac{1}{2}c^2Q_{N_0-1}Q_{N_0}^{-1} \,. \end{aligned} \tag{36}$$

The one-forms $\xi_n^{i\prime}$ are obtained by sorting with respect to the generators X_i.

As was pointed out in [9, 19], discrete symmetries of the kind (33) are preserved by a suitable choice of the zeros λ_r, namely (35). Furthermore, the definition (34) of Q_0 guarantees that the primed version of (32) is satisfied. Thus, we can set

$$\mathcal{X}'_{-1} = X_1\xi_{-1}^{1\prime} \qquad \mathcal{X}'_0 = X_i\xi_0^{i\prime} \qquad \mathcal{X}'_1 = X_i\xi_1^{i\prime}$$

which establishes the Bäcklund transformation for the (ρ, σ) system (4), (5).

On use of the parametrization (6) and (7) the Bäcklund transformation ($N_0 = 1$) of the Maxwell–Bloch system may be phrased as follows: *Let E, ρ, N be a seed solution of the* Maxwell–Bloch system (8*a*). Then a new solution of this system is given by

$$\begin{aligned} &x' = x \qquad t' = t \\ &E' = E - 2\lambda_1 \sin\gamma\ \mathrm{e}^{-\mathrm{i}\beta} \\ &\begin{pmatrix} N' & -\rho' \\ -\rho^{*\prime} & -N' \end{pmatrix} = R(\gamma,\beta)\begin{pmatrix} N & -\rho \\ -\rho^* & -N \end{pmatrix} R(\gamma,\beta) + c^2\lambda_1 R(\gamma,\beta) \end{aligned} \tag{37}$$

where

$$\frac{\phi_1^1}{\phi_1^2} = \alpha\mathrm{e}^{\mathrm{i}\beta} \qquad \alpha = -\tan\frac{\gamma}{2} \qquad R(\gamma,\beta) = \begin{pmatrix} \cos\gamma & \sin\gamma\ \mathrm{e}^{\mathrm{i}\beta} \\ \sin\gamma\ \mathrm{e}^{-\mathrm{i}\beta} & -\cos\gamma \end{pmatrix}$$

and ϕ_1, λ_1 is a solution of the Frobenius system (16)–(18).

Finally, as mentioned at the end of section 2, the Maxwell–Bloch system in the form (11) collapses to the sine–Gordon equation

$$\theta_{xt} = \sin\theta \tag{38}$$

in the limit $c = 0$, E real. In this case, γ obeys the equations

$$\gamma_x = \lambda_1 \sin(\gamma - \theta) \qquad \gamma_t = \theta_t + \lambda_1^{-1}\sin\gamma \tag{39}$$

and the Bäcklund transformation (37) reads

$$\theta' = -\theta + 2\gamma \,. \tag{40}$$

Solving (40) for γ and inserting into (39) we rediscover the classical Bäcklund transformation

$$\left(\frac{\theta'+\theta}{2}\right)_x = \lambda_1 \sin\left(\frac{\theta'-\theta}{2}\right) \qquad \left(\frac{\theta'-\theta}{2}\right)_t = \lambda_1^{-1} \sin\left(\frac{\theta'+\theta}{2}\right)$$

for the sine–Gordon equation (38) [20].

5. The fifth Painlevé equation

This section is devoted to a similarity reduction of the Maxwell–Bloch system. On using suitable similarity variables this will lead to an ordinary differential equation of second order which has been shown to be linked to the fifth Painlevé equation via a contact transformation [10].

Following Winternitz [10], we first solve the Maxwell–Bloch system (8a) for ρ and N so that we obtain a partial differential equation for E only, viz

$$EE_{xtt} - E_{xt}E_t + E^3E_x = \tfrac{1}{2}c^2E^2\,. \tag{41}$$

Here, E has been assumed to be real. Then it is easily seen that the above equation is invariant under the Lie-point symmetry [21]

$$X = 2x\partial_x - t\partial_t + E\partial_E\,.$$

Consequently, equation (41) can be reduced to an ordinary differential equation by introducing similarity variables according to

$$E = \frac{F(z)}{t} \qquad z = t\sqrt{x} \qquad s = \ln x\,.$$

Hence, we end up with the third order equation

$$\left(z\frac{F_{zz}}{F}\right)_z + \frac{F_zF}{z} = c^2\,. \tag{42}$$

Remarkably, the first integral of this equation as given in [10] may be derived directly from the linear problem (16)–(18). To this end we note that in the variables z and s the coefficients of the Lax pair (16), (17) are independent of s iff the integration constant in (19) vanishes, i.e. $\lambda = 1/c\sqrt{x}$. It then assumes the form

$$\phi_z = \left[\frac{1}{2z}\begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix} + \frac{c}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]\phi \tag{43}$$

$$\phi_s = \left[\frac{1}{4c}\begin{pmatrix} zF_{zz}/F & -F_z \\ -F_z & -zF_{zz}/F \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix} - \frac{c}{4}\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}\right]\phi\,. \tag{44}$$

This fact may now be exploited by setting

$$\phi(z,s) = \mathrm{e}^{-\varepsilon s/4}\phi(z)$$

where ε is an arbitrary constant, which transforms (44) into the algebraic system

$$\varepsilon c\phi = \begin{pmatrix} -zF_{zz}/F + c^2z & cF + F_z \\ -cF + F_z & zF_{zz}/F - c^2z \end{pmatrix}\phi\,. \tag{45}$$

Since (45) constitutes an eigenvalue problem, the corresponding determinant has to vanish identically. Thus, the characteristic equation reads

$$z^2\left(F_{zz}/F - c^2\right)^2 + F_z^2 - c^2F^2 = \varepsilon^2c^2\,. \tag{46}$$

It can readily be verified that differentiation of (46) produces the third order equation (42) provided that $F_{zz}/F - c^2 \neq 0$. Finally, the contact transformation

$$F(z) = \mathrm{i}\frac{H - 1 - 2\zeta H_\zeta}{H(H-1)} \qquad z = 2\sqrt{\zeta}$$
$$H(\zeta) = \frac{-\mathrm{i}FF_z + zF_{zz}}{-\mathrm{i}FF_z + z(F_{zz} - c^2 F)} \qquad \zeta = \frac{1}{4}z^2 \tag{47}$$

connects (46) to the Painlevé equation

$$H_{\zeta\zeta} = \left(\frac{1}{2H} + \frac{1}{H-1}\right)H_\zeta^2 - \frac{1}{\zeta}H_\zeta + \frac{(H-1)^2}{8\zeta^2}\left(\varepsilon^2 H - \frac{1}{H}\right) - \frac{c^2}{2\zeta}H\,. \tag{48}$$

Its solution is the fifth Painlevé transcendent $P_V(\frac{1}{2}\varepsilon^2, -\frac{1}{2}, -\frac{1}{2}c^2, 0; \zeta)$. This observation is in agreement with the strong link between the Painlevé property, i.e. the absence of movable singularities other than poles, and integrable partial differential equations [1].

How does the Bäcklund transformation (37) act on the Painlevé equation (48)? For this we remark that the quotient of the two components of the eigenfunction ϕ is obtained from (45) in a purely algebraic manner, namely

$$\alpha = \frac{\phi^1}{\phi^2} = \frac{F_z + cF}{z(F_{zz}/F - c^2) + \varepsilon c}\,.$$

Furthermore, the corresponding 'spatial' part (43) is satisfied modulo (46). Hence, the (Bäcklund) transformation for F is given by

$$F' = F - 2z\frac{F_z^2 - c^2F^2}{z(F_{zz} - c^2F) - \varepsilon F_z}\,. \tag{49}$$

In order to get the transformation law for the constant ε we insert F' into the primed version of (46) and conclude that

$$\pm\,\varepsilon' = \varepsilon + 2\,. \tag{50}$$

The final step in the procedure is to apply the contact transformation (47) to the transformation (49) which yields

$$H' = \frac{(2\zeta H_\zeta + 1 - H^2)^2 - 4\zeta H^2(H-1) - H^2(H-1)^2(\varepsilon+1)^2}{H[(2\zeta H_\zeta + \varepsilon(H-1)^2 - 2(H-1))^2 - 4\zeta H(H-1) - (H-1)^2(\varepsilon+1)^2]} \tag{51}$$

(thanks to MAPLE's factorizer). Here, $c = 1$ has been set without loss of generality.

The transformation properties of the six Painlevé equations have been widely studied by Russian authors. A unified approach and references can be found in [22]. Gromak [23] has given a transformation between the fifth and the third Painlevé equation which we denote by $T_{\mathrm{V}\to\mathrm{III}}$. Moreover, if $P_{\mathrm{III}}(\alpha, \beta, \gamma, \delta; z)$ is a solution of the third Painlevé equation so is $-P_{\mathrm{III}}(-\alpha, -\beta, \gamma, \delta; z)$ (transformation T_{III}). The composite transformation $T_{\mathrm{III}\to\mathrm{V}} \circ T_{\mathrm{III}} \circ T_{\mathrm{V}\to\mathrm{III}}$ then agrees with the one derived in this paper.

To round off we illustrate the fact that the transformation (49) (or (51)) may be iterated and therefore produces chains of solutions by way of an example. On starting with the seed solution

$$F = z \qquad \varepsilon = 1 \qquad c = 1$$

we generate the following hierarchy of rational solutions:

$$F_1 = z$$

$$F_3 = \frac{z(3 - z^2)}{1 + z^2}$$

$$F_5 = \frac{z(z^6 - 9z^4 - 45z^2 + 45)}{z^6 + 3z^4 + 27z^2 + 9} .$$

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References

[1] Levi D and Winternitz P (ed) 1992 *Painlevé Transcendents: Their Asymptotics and Physical Applications* (New York: Plenum)
[2] Steudel H 1991 *Phys. Lett.* **156A** 491
[3] Steudel H 1983 *Physica* **6D** 155
[4] Mikhailov A V and Winternitz P 1993 to be published
[5] Harrison B K 1983 *J. Math. Phys.* **24** 2178
[6] Estabrook F B 1982 *J. Math. Phys.* **23** 2071
[7] Hoenselaers C 1986 *Prog. Theor. Phys.* **75** 1014
[8] Hoenselaers C 1988 *J. Phys. A: Math. Gen.* **21** 17
[9] Hoenselaers C and Schief W K 1992 *J. Phys. A: Math. Gen.* **25** 601
[10] Winternitz P 1992 *Painlevé Transcendents: Their Asymptotics and Physical Applications* ed D Levi and P Winternitz (New York: Plenum)
[11] Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1
[12] Estabrook F B and Wahlquist H D 1976 *J. Math. Phys.* **17** 1293
[13] Kac V G 1985 *Infinite-dimensional Lie Algebras* (Cambridge: Cambridge University Press)
[14] Estabrook F B 1989 *Proc. NATO Advanced Study Institute on Partially Integrable Nonlinear Evolution Equations and their Physical Applications, Les Houches* (Dordrecht: Kluwer)
[15] Cenkl B 1986 *Physica* **18D** 217
[16] Levi D and Sym A 1990 *Phys. Lett.* **149A** 381
[17] Neugebauer G 1979 *J. Phys. A: Math. Gen.* **12** L67
[18] Neugebauer G and Kramer D 1983 *J. Phys. A: Math. Gen.* **16** 1927
[19] Neugebauer G and Meinel R 1984 *Phys. Lett.* **100A** 467
[20] Lamb G L 1976 *Bäcklund Transformations* ed R M Miura (Berlin: Springer)
[21] Stephani H 1989 *Differential Equations: Their Solution Using Symmetries* (Cambridge: Cambridge University Press)
[22] Fokas A S and Ablowitz M J 1982 *J. Math. Phys.* **23** 2033
[23] Gromak V I 1975 *Diff. Urav.* **11** 373